

Some notes on L^p Bernstein inequality when $0 < p < 1$

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Abstract

Recently, Nagy-Toókos and Totik-Varga proved an asymptotically sharp L^p Bernstein type inequality on union of finitely many intervals. We extend this inequality to the case when the power p is between 0 and 1; such sharp Bernstein type inequality was proved first by Arestov.

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1 Introduction and new results

Bernstein inequality from approximation theory is well known. In the last decades it was generalized to arbitrary compact subsets of the real line using potential theoretical quantities. For potential theory, we refer to the books [Ran95] and [ST97]. This general form of Bernstein inequality states that if $K \subset \mathbf{R}$ is a compact set, $x \in K$ interior point, P is an algebraic polynomial, then

$$|P'(x)| \leq \deg(P) \pi \omega_K(x) \|P\|_K \quad (1)$$

where $\omega_K(x)$ is the density of the equilibrium measure. Inequality (1) was proved independently by Baran [Bar92] and Totik [Tot01].

Recently, this inequality was generalized to L^p norms for algebraic polynomials and trigonometric polynomials, see [NT13] and [TV13]. We need the following notation for $E \subset [0, 2\pi)$:

$$\Gamma_E := \{e^{it} : t \in E\}.$$

The following is Theorem 1.1 in [TV13].

Theorem. *Let $1 \leq p < \infty$ and $E \subset [0, 2\pi)$ be a compact set consisting of finitely many intervals. Denote the density of the equilibrium measure of Γ_E by $\omega_{\Gamma_E}(e^{it})$. Then, for any trigonometric polynomial T_n of degree n , we have*

$$\int_E \left| \frac{T_n'(t)}{n 2\pi \omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \leq (1 + o(1)) \int_E |T_n(t)|^p \omega_{\Gamma_E}(e^{it}) dt \quad (2)$$

where $o(1)$ tends to 0 uniformly in T_n as $n \rightarrow \infty$.

In this paper we extend inequality (2) to the case $0 < p < 1$ (Arestov case) and show that the result is asymptotically sharp. That is, we are going to prove the following two theorems.

Theorem 1. *Let $0 < p < 1$ be arbitrary. Let $E \subset [0, 2\pi)$ be a compact set consisting of finitely many intervals. Then, for any trigonometric polynomial T_n of degree n , we have*

$$\int_E \left| \frac{T'_n(t)}{n 2\pi \omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \leq (1 + o(1)) \int_E |T_n(t)|^p \omega_{\Gamma_E}(e^{it}) dt \quad (3)$$

where $o(1)$ tends to 0 uniformly in T_n as $n \rightarrow \infty$.

Theorem 2. *Let $0 < p < 1$ be arbitrary. Let $E \subset [0, 2\pi)$ be a compact set consisting of finitely many intervals. Then, there exist trigonometric polynomials T_n such that*

$$\int_E \left| \frac{T'_n(t)}{n 2\pi \omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \geq (1 - o(1)) \int_E |T_n(t)|^p \omega_{\Gamma_E}(e^{it}) dt$$

where $o(1)$ tends to 0 as $n \rightarrow \infty$.

2 Approach of Theorem 1

The proof of Theorem 1 follows the same idea as that of (2). First, we introduce notations and cite results from the papers [NT13] and [TV13]. Then we discuss the three lemmas which have to be modified and prove them.

We say E is a T-set of order N if there is a real trigonometric polynomial U_N of degree N such that $U_N(t)$ runs through $[-1, 1]$ $2N$ -times as t runs through E , see [TV13], p. 404.

The idea of the proof of (2) consists of three major steps:

- (a) To prove it when E is a so-called T-set associated of the trigonometric polynomial U_N , $E = U_N^{-1}[-1, 1]$ and T_n is a polynomial of U_N ;
- (b) to prove it when E is a T-set and T_n is an arbitrary polynomial;
- (c) to prove it when both the finite interval-system E and the trigonometric polynomial T_n are arbitrary.

To verify (a), it is enough to use Arestov inequality for trigonometric polynomials, see [DL93], Section 4.3 or [Are81] instead of Zygmund inequality, all the other parts of that proof come through.

As for part (c), the proof from [TV13] can be applied here since they did not use the fact that $1 \leq p < \infty$.

As for part (b), most of the ideas from [TV13] (and [NT13]) applies here as well, except for Lemma 3.3 and Proposition 3.5 from [TV13]. To adopt the latters to our case, we need to recall some notations and some details of that approach.

We split the set E as follows (see Section 3 in [TV13]). Let

$$E = \cup_{l=1}^m [v_{2l-1}, v_{2l}] = \cup_{l=1}^m \cup_{h=1}^{r_l} [\zeta_{l,h-1}, \zeta_{l,h}] = \cup_{j=1}^{2N} B_j$$

where $[v_{2l-1}, v_{2l}]$, $l = 1, 2, \dots, m$, denote the components of E and $[\zeta_{l,h-1}, \zeta_{l,h}] = B_{r_1+\dots+r_{l-1}+h}$, $l = 1, 2, \dots, m$, $h = h_l = 1, 2, \dots, r_l$, denote the branches of

E . Recall that a subset of E is called branch if U_N is strictly monotone on this subset, and $U_N(t)$ runs through $[-1, 1]$ precisely once as t runs through this subset. If two branches, B_j and B_{j+1} has a common point, say ζ , then necessarily $|U_N(\zeta)| = 1$ and $U'_N(\zeta) = 0$ and we say that ζ is an inner extremal point.

We fix γ, κ, θ such that

$$1/2 > \theta > 4\kappa$$

and

$$0 < \gamma < \frac{\kappa}{2}.$$

There will be one more assumption. We divide E into intervals I_j of length between $1/2n^\kappa$ and $1/n^\kappa$ as in [TV13], Subsection 3.1. We call such I_j a small interval. J_n denotes the set of indices of the small intervals. Let $J \subset J_n$ be arbitrary. Then $H = H(J)$ denotes the union of $\{I_j\}_{j \in J}$ and H_b denotes the union of the bordering small intervals I_j , for precise definition we refer to Subsection 3.1 in [TV13]. We need the following notations:

$$\begin{aligned} A(T_n, X) &:= \int_X \left| \frac{T'_n(t)}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt, \\ B(T_n, X) &:= \int_X |T_n(t)|^p \omega_{\Gamma_E}(e^{it}) dt, \\ a(T_n, X) &:= \frac{A(T_n, X)}{A(T_n, E)}, \end{aligned}$$

and

$$b(T_n, X) := \frac{B(T_n, X)}{B(T_n, E)}.$$

With these notations a set $H = H(J)$ can possess the following properties:

$$(H \cup H_b) \cap E \subset [\zeta_{l,h}, \zeta_{l,h+1}] \quad (\text{I})$$

for some l and h ;

$$a(T_n, H_b) \leq n^{-\gamma}, \quad (\text{II-a})$$

$$b(T_n, H_b) \leq n^{-\gamma}, \quad (\text{II-b})$$

and

$$|H(J)| \leq 4n^{\gamma-\kappa} = o(1). \quad (\text{III})$$

With this splitting at hand we use another division of E . An interval $H = H(J) \subset E$ can be of first, second or third type. The union of them covers E except at most $4N$ small intervals. For the definition, we refer to [TV13] Subsection 3.2 (Three types of subinterval; see also [NT13] Subsection 5.2). What is important for us is that an interval H of the first or third type possesses the properties (I), (II-a) and (II-b), while an interval H of the second type has the properties (II-a), (II-b) and (III) and there is exactly one $l \in \{1, 2, \dots, m\}$ and exactly one $h = h_l \in \{1, 2, \dots, r_l - 1\}$ such that $\zeta_{l,h_l} \in H$, so an interval of the second type with its bordering small intervals has intersections of positive Lebesgue measures with two branches of E .

First we cite Lukashov's result [Luk04], which extends the trigonometric form of Bernstein-Szegő's inequality from an interval to an arbitrary compact subset of $(-\pi, \pi]$. Actually, we use this following, special case only.

Lemma 3. *Let $E \subset (-\pi, \pi]$ consist of finitely many intervals. If e^{it} is an inner point of Γ_E , then for any trigonometric polynomial T_n of degree at most $n = 1, 2, \dots$ we have*

$$|T'_n(t)| \leq n 2\pi \omega_{\Gamma_E}(e^{it}) \|T_n\|_E. \quad (4)$$

The following lemma states that a T-set E of order N can be deformed such a way that it remains a T-set of order N , see also Proposition 3.5 in [TV13].

Lemma 4. *Let E be the union of the disjoint intervals $[v_{2l-1}, v_{2l}] \subset (-\pi, \pi)$, where $l = 1, 2, \dots, m$ and $v_{2l-1} < v_{2l} < v_{2l+1}$. If E is a T-set of order N , $E = U^{-1}[-1, 1]$, then there is a family of $E(\delta)$, $\delta > 0$ of T-sets of order N such that*

(i)

$$E(\delta) = \bigcup_{l=1}^m [v_{2l-1}, v_{2l}(\delta)],$$

where each $v_{2l}(\delta)$ strictly decreases in δ and converges to v_{2l} for every $l \in \{1, 2, \dots, m\}$ and $E(\delta) \subset E$;

(ii)

if U has the inner extremal points $\zeta_{l,1} < \zeta_{l,2} < \dots < \zeta_{l,r_l-1}$ in its l -th component $[v_{2l-1}, v_{2l}]$ then $E(\delta)$ also has $r_l - 1$ inner extremal points $\zeta_{l,1}(\delta) < \zeta_{l,2}(\delta) < \dots < \zeta_{l,r_l-1}(\delta)$ in $[v_{2l-1}, v_{2l}(\delta)]$ such that each $\zeta_{l,h}(\delta)$ strictly decreases in δ and converges to $\zeta_{l,h}$, $h = h_l \in \{1, 2, \dots, r_l - 1\}$;

(iii)

if ω_{Γ_E} , $\omega_{\Gamma_{E(\delta)}}$ denote the corresponding equilibrium densities of Γ_E and $\Gamma_{E(\delta)}$ then there is a sequence $D_\delta = D(E(\delta)) \rightarrow 1$ for which the estimates

$$1 \leq \frac{\omega_{\Gamma_{E(\delta)}}(e^{it})}{\omega_{\Gamma_E}(e^{it})} \leq D_\delta \quad (5)$$

are valid for every

$$t \in \bigcup_{l=1}^m \left[\frac{v_{2l-1} + \zeta_{l,1}}{2}, \frac{\zeta_{l,r_l-1} + v_{2l}}{2} \right]. \quad (6)$$

and for sufficiently small $\delta > 0$.

Note that if U is strictly monotone on $[v_{2l-1}, v_{2l}]$ for some l , then $r_l = 0$ and we let $\left[\frac{v_{2l-1} + \zeta_{l,1}}{2}, \frac{\zeta_{l,r_l-1} + v_{2l}}{2} \right] := \emptyset$.

We need to approximate the characteristic function of an interval by a trigonometric polynomial of small degree, see Lemma 3.1 in [TV13].

Lemma 5. *Fix $0 < p < \infty$. We know that $1/2 > \theta > 4\kappa$. Then there are constants $C_1, C_2 > 0$ with the following properties. Assume that $H = H(J)$ ($J \subset J_n$) is an interval with characteristic function $\chi_H(t)$. There exists a*

trigonometric polynomial $q = q(H, n; t)$ with $\deg(q) \leq 2n^{2\theta} + 1 \leq 3n^{2\theta}$ which satisfies

$$0 \leq q(t) \leq 1, \quad t \in [-\pi, \pi], \quad (7)$$

furthermore,

$$|q(t) - \chi_H(t)| \leq C_2 e^{-C_1 n^\theta}, \quad (8)$$

$$|q'(t)| \leq C_2 e^{-C_1 n^\theta},$$

whenever $t \in [-\pi, \pi] \setminus H_b$.

The independence of constant C_2 of E and the degree estimate can be seen from following the proof in [NT13].

Let $F_n := C_2 \exp(-C_1 n^\theta)$. Later we will need that there exists $C_3 > 0$ such that

$$F_n^p \leq C_3 n^{-\gamma}. \quad (9)$$

Of course, C_3 depends on γ, θ, C_1 and C_2 only.

There is one more assumption mentioned earlier. This was not present when the power was greater than 1 (here $p \leq 1$). The assumption is

$$(1 - 2\theta)p \geq \gamma. \quad (\text{IV})$$

For example, $\theta = 1/4$, $\kappa = 1/32$ and $\gamma = \min(1/65, p/2)$ is a good choice.

Next lemma says that the integral of a trigonometric polynomial cannot be arbitrarily small, see Lemma 3.12 in [TV13]. This is a Nikolskii type inequality.

Lemma 6. *Let $0 < p < \infty$, E consist of finitely many intervals, I be a fixed subinterval of E and let T be an arbitrary trigonometric polynomial with the property $\sup_{t \in I} |T(t)| = 1$. Then, there exists $C_4 > 0$ depending on the length of I only (and is independent of E, p and T) such that*

$$\int_I |T(t)|^p \omega_{\Gamma_E}(e^{it}) dt \geq C_4 \frac{1}{2^p} \frac{1}{(\deg T)^2}.$$

The following lemma is a symmetrization technique for trigonometric polynomials. For proof and details, we refer to [TV13], see Lemma 3.2.

Lemma 7. *Let E be a T -set associated with the trigonometric polynomial U_N of degree N . For a point $t \in E$ with $U_N(t) \in (-1, 1)$ let t_1, t_2, \dots, t_{2N} be those points in E which satisfy $U_N(t_h) = U_N(t)$. If V_n is a trigonometric polynomial of degree at most n , then there is an algebraic polynomial $S_{n/N}$ of degree at most n/N such that*

$$\sum_{h=1}^{2N} V_n(t_h) = S_{[n/N]}(U_N(t)).$$

If $U_{N,h}^{-1}$ denotes the inverse of U_N restricted on the branch B_h then $t_h = U_{N,h}^{-1}(U_N(t))$. This shows that

$$\frac{d}{dt} t_h = \frac{U'_N(t)}{U'_N(t_h)}. \quad (10)$$

It is known (see, e.g., Lemma 3.1 in [Tot12]) that

$$\omega_{\Gamma_E}(e^{it}) = \frac{1}{2\pi N} \frac{|U'_N(t)|}{\sqrt{1 - U_N(t)^2}}. \quad (11)$$

The previous equation, together with (10), shows that

$$\left| \frac{\omega_{\Gamma}(e^{it_h}) \frac{d}{dt} t_h}{\omega_{\Gamma}(e^{it})} \right| = 1. \quad (12)$$

Let T_n be a trigonometric polynomial of degree n , and let q be the trigonometric polynomial from Lemma 5. Then, by the previous lemma,

$$T_n^*(t) := \sum_{h=1}^{2N} T_n(t_h) q(t_h)$$

is a polynomial of the trigonometric polynomial U_N , so T_n^* makes a link between step (a) and step (b). This connection is shown by the the analogue Lemmas 8 and 9 of Lemma 3.3 from [TV13] (see also Lemmas 7, 8 in [NT13]). We will prove the following three lemmas in the next subsection.

Lemma 8. *Let $0 < p < 1$. Suppose we have a T -set E associated with the trigonometric polynomial U_N of degree N . We also have an interval $H = H(J)$ satisfying the property (I). Then, using T_n^* defined for T_n and $n^* := \deg(T_n^*) \leq \deg(T_n) + \deg(q)$, we have*

$$\begin{aligned} & \left| \left(\frac{n^*}{n} \right)^p A(T_n^*, E) - 2N A(T_n, H) \right| \\ & \leq 2N (4F_n^p + a(T_n, H_b)) A(T_n, E) + (2N) 4 \cdot 3^p \left(\frac{n^{2\theta}}{n} \right)^p B(T_n, E). \end{aligned}$$

Lemma 9. *With the same assumption as in Lemma 8 we also have that*

$$|B(T_n^*, E) - 2N B(T_n, H)| \leq 2N (3F_n^p + b(T_n, H_b)) B(T_n, E).$$

Analogously to $A(T_n, X)$ and $B(T_n, X)$ we use the notations $A_\delta(T_n, X)$ and $B_\delta(T_n, X)$:

$$A_\delta(T_n, X) = \int_X \left| \frac{T_n'(t)}{n 2\pi \omega_{\Gamma_{E(\delta)}}(e^{it})} \right|^p \omega_{\Gamma_{E(\delta)}}(e^{it}) dt,$$

and

$$B_\delta(T_n, X) = \int_X |T_n(t)|^p \omega_{\Gamma_{E(\delta)}}(e^{it}) dt$$

respectively, where $E(\delta)$ comes from Lemma 4.

Next lemma corresponds to Proposition 3.5 from [TV13].

Lemma 10. *Let $0 < p < 1$. Assume that H is an interval of second type. Let $q = q(H, n; t)$ be the polynomial from Lemma 5 and let X be an arbitrary subset of E . Then the following estimates hold:*

$$\begin{aligned} |A(T_n q, H) - A(T_n, H)| \\ \leq \left(F_n^p + 3^p \left(\frac{n^{2\theta}}{n} \right)^p \right) A(T_n, E) + 3^p \left(\frac{n^{2\theta}}{n} \right)^p B(T_n, E), \end{aligned} \quad (13)$$

$$A(T_n q, X) \leq A(T_n, X) + 3^p \left(\frac{n^{2\theta}}{n} \right)^p B(T_n, E), \quad (14)$$

$$B(T_n q, X) \leq B(T_n, X). \quad (15)$$

We also have, if n is large,

$$\begin{aligned} |A_\delta(T_n q, E(\delta)) - A_\delta(T_n q, H)| &\leq D_\delta^{1-p} a(T_n, H_b) A(T_n, E) \\ &+ \left(D_\delta^{1-p} F_n^{p/2} + D_\delta^{1-p} 3^p \left(\frac{n^{2\theta}}{n} \right)^p + \left(\frac{n^{2\theta}}{n} \right)^p b(T_n, H_b) \right) B(T_n, E), \end{aligned} \quad (16)$$

and

$$|B_\delta(T_n q, E(\delta)) - B_\delta(T_n q, H)| \leq D_\delta (F_n^p + b(T_n, H_b)) B(T_n, E). \quad (17)$$

2.1 Proofs of the Lemmas 8, 9 and 10

2.1.1 Proofs of the Lemmas

Interestingly, the proofs of Lemmas 8 and 9 are somewhat simpler than those of in Nagy-Toókos [NT13] and Totik-Varga [TV13]. This „simplicity” derives from the following inequalities:

$$||a|^p - |b|^p| \leq |a - b|^p \quad (18)$$

$$|a + b|^p \leq |a|^p + |b|^p, \quad (19)$$

where $a, b \in \mathbb{R}$ and $0 < p < 1$.

During the verification of Lemmas 8 and 9 we frequently use the subsequent identities (cf. formulas (61), (62) in [NT13]). Let X be a subset of the branch B_{h_0} and let X_h denote the set $U_{N,h}^{-1}(U_N(X))$. Then

$$\int_X f(t_h) \omega_{\Gamma_E}(e^{it}) dt = \int_{X_h} f(t) \omega_{\Gamma_E}(e^{it}) dt, \quad (20)$$

and

$$\int_X \left(\frac{(f(t_h))'}{\omega_{\Gamma_E}(e^{it})} \right)^p \omega_{\Gamma_E}(e^{it}) dt = \int_{X_h} \left(\frac{(f(t))'}{\omega_{\Gamma_E}(e^{it})} \right)^p \omega_{\Gamma_E}(e^{it}) dt \quad (21)$$

respectively.

Proof of Lemma 8.

$$\begin{aligned}
& \left| \left(\frac{n^*}{n} \right)^p A(T_n^*, E) - 2NA(T_n, H) \right| \\
& \leq \left| \left(\frac{n^*}{n} \right)^p A(T_n^*, E \setminus U_N^{-1}(U_N(H \cup H_b))) \right| + \left| \left(\frac{n^*}{n} \right)^p A(T_n^*, U_N^{-1}(U_N(H_b))) \right| \\
& \quad + \left| \left(\frac{n^*}{n} \right)^p A(T_n^*, U_N^{-1}(U_N(H))) - 2NA(T_n, H) \right| \quad (22)
\end{aligned}$$

We separately estimate the terms of the right hand side. We begin with the first term. By (19), we get

$$\begin{aligned}
& \left(\frac{n^*}{n} \right)^p A(T_n^*, E \setminus U_N^{-1}(U_N(H \cup H_b))) \\
& \leq \sum_{h=1}^{2N} \int_{E \setminus U_N^{-1}(U_N(H \cup H_b))} \left| \frac{(T_n(t_h))' q(t_h)}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\
& \quad + \sum_{h=1}^{2N} \int_{E \setminus U_N^{-1}(U_N(H \cup H_b))} \left| \frac{T_n(t_h)(q(t_h))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt. \quad (23)
\end{aligned}$$

On $E \setminus U_N^{-1}(U_N(H \cup H_b))$ we know that $t_h \notin H \cup H_b$, therefore by (8), $|q(t_h)| \leq C_2 e^{-C_1 n^\theta} = F_n$. This fact is applied to the first term of the right hand side of (23). As regards the second term, we use Lukashov's inequality (4) and (12). Then increasing the domain of integration $E \setminus U_N^{-1}(U_N(H \cup H_b))$ to $E = \cup_{j=1}^{2N} B_j$ the previous inequality is continued as

$$\begin{aligned}
& \leq F_n^p \sum_{h=1}^{2N} \sum_{j=1}^{2N} \int_{B_j} \left| \frac{(T_n(t_h))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\
& \quad + \left(\frac{\deg(q)}{n} \right)^p \sum_{h=1}^{2N} \sum_{j=1}^{2N} \int_{B_j} |T_n(t_h)|^p \omega_{\Gamma_E}(e^{it}) dt.
\end{aligned}$$

By (20) and (21) we get

$$\begin{aligned}
& \leq 2NF_n^p \sum_{h=1}^{2N} \int_{B_h} \left| \frac{(T_n(t))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\
& \quad + 2N \left(\frac{\deg(q)}{n} \right)^p \sum_{h=1}^{2N} \int_{B_h} |T_n(t)|^p \omega_{\Gamma_E}(e^{it}) dt \\
& \quad = o(1)A(T_n, E) + o(1)B(T_n, E),
\end{aligned}$$

remembering the facts that $F_n = o(1)$ and $\deg(q)/n = o(1)$. That is we have

$$\begin{aligned}
& \left(\frac{n^*}{n} \right)^p A(T_n^*, E \setminus U_N^{-1}(U_N(H \cup H_b))) \\
& \leq (2N) F_n^p A(T_n, E) + (2N) \left(\frac{3n^{2\theta}}{n} \right)^p B(T_n, E) \quad (24)
\end{aligned}$$

or, without detailing the error terms, we can write

$$\left(\frac{n^*}{n}\right)^p A(T_n^*, E \setminus U_N^{-1}[U_N(H \cup H_b)]) \leq o(1) A(T_n, E) + o(1) B(T_n, E). \quad (25)$$

Now we turn to the second term of the right hand side of (22). Denote by h_0 the index of the branch which $(H \cup H_b) \cap E$ belongs to (such index exists because of the property (I)). By (19) we obtain

$$\begin{aligned} & \left(\frac{n^*}{n}\right)^p A(T_n^*, U_N^{-1}[U_N(H_b)]) \\ & \leq \int_{U_N^{-1}(U_N(H_b))} \left| \frac{(T_n(t_{h_0}))' q(t_{h_0})}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & + \sum_{h \neq h_0} \int_{U_N^{-1}(U_N(H_b))} \left| \frac{(T_n(t_h))' q(t_h)}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & + \int_{U_N^{-1}(U_N(H_b))} \left| \frac{T_n(t_{h_0})(q(t_{h_0}))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & + \sum_{h \neq h_0} \int_{U_N^{-1}(U_N(H_b))} \left| \frac{T_n(t_h)(q(t_h))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt. \quad (26) \end{aligned}$$

The estimations of the second and the fourth term of the right hand side of (26) follow that of the first term at the right hand side of (22). If $h \neq h_0$ then (8) again implies that $|q(t_h)| \leq F_n$ on the set $U_N^{-1}[U_N(H_b)]$, therefore, for these two terms, the subsequent inequality holds:

$$\begin{aligned} & \sum_{h \neq h_0} \int_{U_N^{-1}(U_N(H_b))} \left| \frac{(T_n(t_h))' q(t_h)}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & + \sum_{h \neq h_0} \int_{U_N^{-1}(U_N(H_b))} \left| \frac{T_n(t_h)(q(t_h))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \leq 2NF_n^p \sum_{h \neq h_0} \int_{B_h} \left| \frac{(T_n(t))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & + 2N \left(\frac{\deg(q)}{n} \right)^p \sum_{h \neq h_0} \int_{B_h} |T_n(t)|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \leq o(1)A(T_n, E) + o(1)B(T_n, E). \quad (27) \end{aligned}$$

For the first term of the right hand side of (26) we get, by $\|q\|_E \leq 1$ and by (21), that

$$\begin{aligned} & \int_{U_N^{-1}(U_N(H_b))} \left| \frac{(T_n(t_{h_0}))' q(t_{h_0})}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \leq \sum_{j=1}^{2N} \int_{((H_b)_j)_{h_0}} \left| \frac{T_n'(t)}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt. \end{aligned}$$

Since $(H_b)_{h_0} = H_b$ (where, by notational conventions, $(H_b)_{h_0} = U_{N,h_0}^{-1} [U_N (H_b)]$), and $((H_b)_j)_{h_0} = (H_b)_{h_0} = H_b$, we can continue as

$$= 2NA(T_n, H_b) = 2Na(T_n, H_b)A(T_n, E). \quad (28)$$

In order to estimate the third term of the right hand side of (26) we use Lukashov's inequality (4), (20) and (12). Then

$$\begin{aligned} & \int_{U_N^{-1}(U_N(H_b))} \left| \frac{T_n(t_{h_0})(q(t_{h_0}))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \leq \sum_{j=1}^{2N} \int_{(H_b)_j} \left| \frac{T_n(t_{h_0}) \deg(q) 2\pi\omega_{\Gamma_E}(e^{it_{h_0}}) \frac{d}{dt} t_{h_0}}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & = 2N \left(\frac{\deg(q)}{n} \right)^p \int_{H_b} |T_n(t)|^p \omega_{\Gamma_E}(e^{it}) dt \leq o(1) B(T_n, E). \end{aligned}$$

This, together with (26), (27) and (28), implies that

$$\begin{aligned} & \left(\frac{n^*}{n} \right)^p A(T_n^*, U_N^{-1}[U_N(H_b)]) \\ & \leq ((2N)F_n^p + 2Na(T_n, H_b)A(T_n, E)) + 2(2N) \left(\frac{3n^{2\theta}}{n} \right)^p B(T_n, E) \quad (29) \end{aligned}$$

or, without detailing the error terms, we can write

$$\left(\frac{n^*}{n} \right)^p A(T_n^*, U_N^{-1}[U_N(H_b)]) \leq (o(1) + 2Na(T_n, H_b))A(T_n, E) + o(1)B(T_n, E), \quad (30)$$

where $o(1)$ is independent of T_n .

As regards the third term of the right hand side of the inequality (22), consider, by (21), that

$$2NA(T_n, H) = \int_{U_N^{-1}(U_N(H))} \left| \frac{T_n'(t_{h_0})}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt,$$

so, by (18), then by (19), we obtain

$$\begin{aligned} & \left| \left(\frac{n^*}{n} \right)^p A(T_n^*, U_N^{-1}(U_N(H))) - 2NA(T_n, H) \right| \\ & \leq \int_{U_N^{-1}(U_N(H))} \left| \frac{(T_n^*(t))' - (T_n(t_{h_0}))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \leq \sum_{h \neq h_0} \int_{U_N^{-1}(U_N(H))} \left| \frac{(T_n(t_h))' q(t_h)}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \quad + \sum_{h=1}^{2N} \int_{U_N^{-1}(U_N(H))} \left| \frac{T_n(t_h)(q(t_h))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \quad + \int_{U_N^{-1}(U_N(H))} \left| \frac{(T_n(t_{h_0}))' q(t_{h_0}) - (T_n(t_{h_0}))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt. \quad (31) \end{aligned}$$

For the first term of the right hand side we again use that, by (8), $|q(t_h)| \leq F_n$ on the set $U_N^{-1}[U_N(H)]$ for $h \neq h_0$. Hence, considering (21), we obtain

$$\begin{aligned} & \sum_{h \neq h_0} \int_{U_N^{-1}(U_N(H))} \left| \frac{(T_n(t_h))' q(t_h)}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \leq F_n^p \sum_{h \neq h_0} \sum_{j=1}^{2N} \int_{((H)_j)_h} \left| \frac{T'_n(t)}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & = F_n^p 2N \int_{\cup_{h \neq h_0} (H)_h} \left| \frac{T'_n(t)}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \leq (2N) F_n^p A(T_n, E) = o(1) A(T_n, E) \quad (32) \end{aligned}$$

where $((H)_j)_h = U_{N,h}^{-1} \left[U \left(U_{N,j}^{-1} [U_N(H)] \right) \right] = (H)_h$. The second term of the right hand side of (31) is estimated by the help of Lukashov's inequality (4). We get

$$\begin{aligned} & \sum_{h=1}^{2N} \int_{U_N^{-1}(U_N(H))} \left| \frac{T_n(t_h) (q(t_h))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \leq \sum_{h=1}^{2N} \int_{U_N^{-1}(U_N(H))} \left| \frac{T_n(t_h) \deg(q) 2\pi q(t_h) \omega_{\Gamma_E}(e^{it_h}) \frac{d}{dt} t_h}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \end{aligned}$$

This inequality is continued by (12) and (20) as

$$\begin{aligned} & \leq \left(\frac{\deg(q)}{n} \right)^p \sum_{h=1}^{2N} \sum_{j=1}^{2N} \int_{((H)_j)_h} |T_n(t)|^p \omega_{\Gamma_E}(e^{it}) dt \\ & = \left(\frac{\deg(q)}{n} \right)^p 2N \int_{\cup_{h=1}^{2N} (H)_h} |T_n(t)|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \leq \left(\frac{3n^{2\theta}}{n} \right)^p (2N) B(T_n, E) = o(1) B(T_n, E). \quad (33) \end{aligned}$$

By (8) we have on the set $U_N^{-1}[U_N(H)]$ that $|q(t_{h_0}) - 1| \leq F_n$. Hence the third term of the right hand side of the inequality (31) can be estimated as

$$\begin{aligned} & \int_{U_N^{-1}(U_N(H))} \left| \frac{(T_n(t_{h_0}))' q(t_{h_0}) - (T_n(t_{h_0}))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ & = \int_{U_N^{-1}(U_N(H))} \left| \frac{(T_n(t_{h_0}))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p |q(t_{h_0}) - 1|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \leq F_n^p \int_{U_N^{-1}(U_N(H))} \left| \frac{(T_n(t_{h_0}))'}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt. \end{aligned}$$

By (21) we continue the inequality as

$$= F_n^p 2N \int_H \left| \frac{T'_n(t)}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \leq o(1) A(T_n, E). \quad (34)$$

So by (31), (32), (33) and (34) we get that

$$\begin{aligned} & \left| \left(\frac{n^*}{n} \right)^p A(T_n^*, U_N^{-1}[U_N(H)]) - 2NA(T_n, H) \right| \\ & \leq (2F_n^p 2N) A(T_n, E) + \left(\frac{3n^{2\theta}}{n} \right)^p (2N) B(T_n, E). \end{aligned} \quad (35)$$

Now combining (22), (24), (29) and (35), we can write

$$\begin{aligned} & \left| \left(\frac{n^*}{n} \right)^p A(T_n^*, E) - 2NA(T_n, H) \right| \\ & \leq (2N) (4F_n^p + a(T_n, H_b)) A(T_n, E) \\ & \quad + 4(2N) \left(\frac{3n^{2\theta}}{n} \right)^p B(T_n, E). \end{aligned}$$

This way we proved the lemma. \square

Proof of Lemma 9. The proof is sketched briefly, because it is very similar to that of Lemma 8. We again split E into three sets: $E \setminus U_N^{-1}[U_N(H \cup H_b)]$, $U_N^{-1}[U_N(H_b)]$ and $U_N^{-1}[U_N(H)]$ and start with the inequality

$$\begin{aligned} & |B(T_n^*, E) - (2N)B(T_n, H)| \\ & \leq |B(T_n^*, E \setminus U_N^{-1}[U_N(H \cup H_b)])| \\ & \quad + |B(T_n^*, U_N^{-1}[U_N(H_b)])| \\ & \quad + |B(T_n^*, U_N^{-1}[U_N(H)]) - (2N)B(T_n, H)|. \end{aligned} \quad (36)$$

Now, as before, we separately estimate the three terms at the right hand side of the inequality.

In the case of the first term we first use (8) and (20) then $E \setminus U_N^{-1}[U_N(H \cup H_b)]$ is increased to E . We get that

$$\begin{aligned} |B(T_n^*, E \setminus U_N^{-1}[U_N(H \cup H_b)])| & \leq F_n^p 2N \int_{\cup_{h=1}^{2N} B_h} |T_n(t)|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \leq o(1)B(T_n, E). \end{aligned} \quad (37)$$

In the case of the second and the third terms we again deal with the h_0 -th term of T_n^* separately and apply (20) and (8). Then, for the second term, we have

$$\begin{aligned} & B(T_n^*, U_N^{-1}[U_N(H_b)]) \\ & \leq F_n^p \sum_{h \neq h_0} \sum_{j=1}^{2N} \int_{((H_b)_j)_h} |T_n(t)|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \quad + 2N \int_{H_b} |T_n(t)|^p \omega_{\Gamma_E}(e^{it}) dt \\ & \leq F_n^p 2N B(T_n, E) + 2Nb(T_n, H_b) B(T_n, E). \end{aligned} \quad (38)$$

As regards the third term, we estimate similarly as in (31) and we use (18) and (19) here. This way we obtain

$$\begin{aligned}
& |B(T_n^*, U_N^{-1}[U_N(H)]) - 2NB(T_n, H)| \\
& \leq F_n^p 2N \sum_{h \neq h_0} \int_{((H)_j)_h} |T_n(t)|^p \omega_{\Gamma_E}(e^{it}) dt \\
& \quad + 2N \int_H |T_n(t)|^p |q(t) - 1|^p \omega_{\Gamma_E}(e^{it}) dt \\
& \leq F_n^p 2N \sum_{h=1}^{2N} \int_{(H)_h} |T_n(t)|^p \omega_{\Gamma_E}(e^{it}) dt \leq F_n^p 2NB(T_n, E). \quad (39)
\end{aligned}$$

Using the inequalities (36), (37), (38) and (39), we have

$$|B(T_n^*, E) - (2N)B(T_n, H)| \leq 2N(3F_n^p + b(T_n, H_b))B(T_n, E).$$

This way we proved the lemma. \square

Proof of Lemma 10. We begin with the verification of (13). (18) and (19) imply that

$$\begin{aligned}
& \left| \int_H \left| \frac{(T_n(t)q(t))'}{\deg(T_nq)2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt - \int_H \left| \frac{T_n'(t)}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \right| \\
& \leq \int_H \left| \frac{\frac{n}{\deg(T_nq)}T_n'(t)q(t) - T_n'(t)}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\
& \quad + \int_H \left| \frac{T_n(t)q'(t)}{\deg(T_nq)2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt.
\end{aligned}$$

Using Lukashov's inequality (4) we obtain that

$$\begin{aligned}
& \leq \int_H \left| \frac{T_n'(t)}{n2\pi\omega_{\Gamma_E}(e^{it})} \left(\frac{n}{\deg(T_nq)}q(t) - 1 \right) \right|^p \omega_{\Gamma_E}(e^{it}) dt \\
& \quad + \int_H \left| \frac{T_n(t)\deg(q)2\pi\omega_{\Gamma_E}(e^{it})\|q\|_E}{\deg(T_nq)2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\
& \leq A(T_n, H) \left(F_n^p + \left| \frac{n}{n+3n^{2\theta}} - 1 \right|^p \right) + B(T_n, H) \left(\frac{\deg q}{\deg(T_nq)} \right)^p \\
& \leq A(T_n, E) \left(F_n^p + \left(\frac{3n^{2\theta}}{n} \right)^p \right) + B(T_n, E) \left(\frac{3n^{2\theta}}{n} \right)^p.
\end{aligned}$$

This way we established (13).

For (14),

$$\begin{aligned}
A(T_nq, X) & \leq \int_X \left| \frac{T_n'(t)q(t)}{\deg(T_nq)2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\
& \quad + \int_X \left| \frac{T_n(t)q'(t)}{\deg(T_nq)2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt
\end{aligned}$$

Apply (7) to the first integral and Lukashov's inequality (4) to the second one to continue the inequality as

$$\begin{aligned} &\leq \left(\frac{n}{\deg(T_n q)} \right)^p A(T_n, X) + \left(\frac{\deg(q)}{\deg(T_n q)} \right)^p B(T_n, X) \\ &\leq A(T_n, X) + \left(\frac{3n^{2\theta}}{n} \right)^p B(T_n, X) \end{aligned}$$

where we again use (IV). This proves (14).

(15) is immediate consequence of (7).

Before the verification of (16) and (17), recall that H is an interval of the second type. Hence, for sufficiently large n , H is subset of the union in (6). This comes from property (III) and that H contains precisely one inner extremal point. For (16), we have

$$|A_\delta(T_n q, E(\delta)) - A_\delta(T_n q, H)| = A_\delta(T_n q, H_b) + A_\delta(T_n q, E(\delta) \setminus (H \cup H_b)).$$

First we estimate $A_\delta(T_n q, H_b)$. (19) shows that

$$\begin{aligned} A_\delta(T_n q, H_b) &= \int_{H_b} \left| \frac{(T_n(t) q(t))'}{\deg(T_n q) 2\pi \omega_{\Gamma_{E(\delta)}}(e^{it})} \right|^p \omega_{\Gamma_{E(\delta)}}(e^{it}) dt \\ &\leq \int_{H_b} \left| \frac{T'_n(t) q(t)}{\deg(T_n q) 2\pi \omega_{\Gamma_{E(\delta)}}(e^{it})} \right|^p \omega_{\Gamma_{E(\delta)}}(e^{it}) dt \\ &\quad + \int_{H_b} \left| \frac{T_n(t) q'(t)}{\deg(T_n q) 2\pi \omega_{\Gamma_{E(\delta)}}(e^{it})} \right|^p \omega_{\Gamma_{E(\delta)}}(e^{it}) dt \end{aligned}$$

First employ (5) to both integrals to replace $\omega_{\Gamma_{E(\delta)}}$ with ω_{Γ_E} and then we apply Lukashov's inequality (4). Then the inequality is continued as

$$\begin{aligned} &\leq D_\delta^{1-p} \int_{H_b} \left| \frac{T'_n(t) q(t)}{\deg(T_n q) 2\pi \omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ &\quad + D_\delta^{1-p} \int_{H_b} \left| \frac{T_n(t) q'(t)}{\deg(T_n q) 2\pi \omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it}) dt \\ &\leq D_\delta^{1-p} \left(\frac{n}{\deg(T_n q)} \right)^p a(T_n, H_b) A(T_n, E) \\ &\quad + D_\delta^{1-p} \left(\frac{\deg(q)}{\deg(T_n q)} \right)^p b(T_n, H_b) B(T_n, E) \\ &\leq D_\delta^{1-p} a(T_n, H_b) A(T_n, E) + D_\delta^{1-p} 3^p \left(\frac{n^{2\theta}}{n} \right)^p b(T_n, H_b) B(T_n, E). \end{aligned}$$

Now we turn to $A_\delta(T_n q, E(\delta) \setminus (H \cup H_b))$. We may assume that $\|T_n\|_E = 1$. (19) gives that

$$\begin{aligned}
& A_\delta(T_n q, E(\delta) \setminus (H \cup H_b)) \\
& \leq \int_{E(\delta) \setminus (H \cup H_b)} \left| \frac{T'_n(t) q(t)}{\deg(T_n q) 2\pi \omega_{\Gamma_{E(\delta)}}(e^{it})} \right|^p \omega_{\Gamma_{E(\delta)}}(e^{it}) dt \\
& \quad + \int_{E(\delta) \setminus (H \cup H_b)} \left| \frac{T_n(t) q'(t)}{\deg(T_n q) 2\pi \omega_{\Gamma_{E(\delta)}}(e^{it})} \right|^p \omega_{\Gamma_{E(\delta)}}(e^{it}) dt.
\end{aligned}$$

By (5), (8) and Lukashov's inequality (4) we can continue as

$$\begin{aligned}
& \leq D_\delta^{1-p} F_n^p \int_{E \setminus (H \cup H_b)} \left| \frac{n 2\pi \|T_n\|_{E(\delta)} \omega_{\Gamma_{E(\delta)}}(e^{it})}{\deg(T_n q) 2\pi \omega_{\Gamma_{E(\delta)}}(e^{it})} \right|^p \omega_{\Gamma_{E(\delta)}}(e^{it}) dt \\
& \quad + D_\delta^{1-p} \int_{E \setminus (H \cup H_b)} \left| \frac{T_n(t) \deg(q) 2\pi \|q\|_{E(\delta)} \omega_{\Gamma_{E(\delta)}}(e^{it})}{\deg(T_n q) 2\pi \omega_{\Gamma_{E(\delta)}}(e^{it})} \right|^p \omega_{\Gamma_{E(\delta)}}(e^{it}) dt \\
& \leq D_\delta^{1-p} F_n^p \left(\frac{n}{\deg(T_n q)} \right)^p \int_{E \setminus (H \cup H_b)} 1 \omega_{\Gamma_{E(\delta)}}(e^{it}) dt \\
& \quad + D_\delta^{1-p} \left(\frac{\deg(q)}{\deg(T_n q)} \right)^p \int_{E \setminus (H \cup H_b)} |T_n(t)|^p \omega_{\Gamma_{E(\delta)}}(e^{it}) dt \\
& \leq D_\delta^{1-p} F_n^p \left(\frac{n}{\deg(T_n q)} \right)^p + D_\delta^{1-p} \left(\frac{\deg(q)}{\deg(T_n q)} \right)^p B(T_n, E).
\end{aligned}$$

The last expression is less than $o(1)B(T_n, E)$, if we consider that $F_n^{p/2} \leq B(T_n, E)$ where n is large enough. This is implied by Lemma 6 as follows. Assume that $x_0 \in E$ is a point for which $|T_n(x_0)| = \|T_n\|_E = 1$. Then x_0 is also an element of a component $[v_{2l-1}, v_{2l}]$ for some l . Now by Lemma 6 we have

$$\int_{v_{2l-1}}^{v_{2l}} |T_n(t)|^p \omega_{\Gamma_E}(t) dt \geq C_4 \frac{1}{2^p} \frac{1}{n^2}.$$

If n is large enough depending on C_4 , p (and F_n), then we can write

$$C_4 \frac{1}{2^p} \frac{1}{n^2} \geq F_n^{p/2} = C_2^{p/2} \exp\left(-C_1 n^\theta \cdot \frac{p}{2}\right). \quad (40)$$

where the second inequality holds if n is large enough. Using these, we can continue the estimate

$$D_\delta^{1-p} F_n^p \left(\frac{n}{\deg(T_n q)} \right)^p \leq D_\delta^{1-p} F_n^{p/2} F_n^{p/2} \leq D_\delta^{1-p} F_n^{p/2} B(T_n, E)$$

and

$$D_\delta^{1-p} \left(\frac{\deg(q)}{\deg(T_n q)} \right)^p B(T_n, E) \leq D_\delta^{1-p} 3^p \left(\frac{n^{2\theta}}{n} \right)^p B(T_n, E).$$

Therefore

$$A_\delta(T_n q, E(\delta) \setminus (H \cup H_b)) \leq D_\delta^{1-p} \left(F_n^{p/2} + 3^p \left(\frac{n^{2\theta}}{n} \right)^p \right) B(T_n, E).$$

Finally, summing up, we can write

$$|A_\delta(T_n q, E(\delta)) - A_\delta(T_n q, H)| \leq D_\delta^{1-p} a(T_n, H_b) A(T_n, E) \\ + \left(D_\delta^{1-p} F_n^{p/2} + D_\delta^{1-p} 3^p \left(\frac{n^{2\theta}}{n} \right)^p + \left(\frac{n^{2\theta}}{n} \right)^p b(T_n, H_b) \right) B(T_n, E).$$

The proof of (17) is similar to that of (16) but it is simpler. First

$$|B_\delta(T_n q, E(\delta)) - B_\delta(T_n q, H)| = B_\delta(T_n q, H_b) + B_\delta(T_n q, E(\delta) \setminus (H \cup H_b)).$$

Then

$$B_\delta(T_n q, H_b) = \int_{H_b} |T_n(t) q(t)|^p \omega_{\Gamma_{E(\delta)}}(e^{it}) dt \\ \leq D_\delta \int_{H_b} |T_n(t) q(t)|^p \omega_{\Gamma_E}(e^{it}) dt \\ \leq D_\delta b(T_n, H_b) B(T_n, E)$$

and

$$B_\delta(T_n q, E(\delta) \setminus (H \cup H_b)) = \int_{E(\delta) \setminus (H \cup H_b)} |T_n(t) q(t)|^p \omega_{\Gamma_{E(\delta)}}(e^{it}) dt \\ \leq D_\delta F_n^p B_\delta(T_n, E(\delta) \setminus (H \cup H_b)) \leq D_\delta F_n^p B(T_n, E).$$

Finally, summing up, we can write

$$|B_\delta(T_n q, E(\delta)) - B_\delta(T_n q, H)| \leq D_\delta (F_n^p + b(T_n, H_b)) B(T_n, E).$$

□

2.2 Application of lemmas

Here we apply the previous lemmas (see [TV13], pp. 401-416 and see that of [NT13]) and investigate the occurring error terms in detail.

2.3 Reviewing the first and third cases

First, we consider Subsection 5.3 from [NT13] and Subsection 3.3 from [TV13]. This setting is called first and third cases. Recall that the set $H \subset E$ satisfies (I), (II-a) and (II-b). We follow those steps, see (30) in [NT13] and (17) in [TV13]. Therefore, with Lemmas 8 and 9, we can write

$$(2N) A(T_n, H) \leq (2N) \left(\frac{\deg(T_n q)}{n} \right)^p B(T_n, H) \\ + 2N (4F_n^p + a(T_n, H_b)) A(T_n, E) \\ + 2N \left(4 \cdot 3^p \left(\frac{n^{2\theta}}{n} \right)^p + 3 \left(\frac{n^*}{n} \right)^p F_n^p + \left(\frac{n^*}{n} \right)^p b(T_n, H_b) \right) B(T_n, E)$$

We can rewrite it with the help of

$$\left(\frac{\deg(T_n q)}{n} \right)^p - 1 = \left(1 + \frac{\deg q}{n} \right)^p - 1 \leq \left(1 + \frac{3n^{2\theta}}{n} \right)^p - 1 \leq p \frac{3n^{2\theta}}{n}$$

where we used $0 < p < 1$.

Here, using Lemma 8, (9) and (II-a), the error term of $A(T_n, E)$ can be estimated as follows

$$4F_n^p + a(T_n, H_b) \leq \frac{4C_3 + 1}{n^\gamma}$$

and, for the error term of $B(T_n, E)$,

$$\begin{aligned} & 4 \cdot 3^p \left(\frac{n^{2\theta}}{n} \right)^p + \left(\frac{n^*}{n} \right)^p F_n^p + \left(\frac{n^*}{n} \right)^p b(T_n, H_b) + p \frac{3n^{2\theta}}{n} \\ & \leq 4 \cdot 3^p \frac{1}{n^\gamma} + 3 \cdot 2^p C_3 \frac{1}{n^\gamma} + 2^p \frac{1}{n^\gamma} + 3p \left(\frac{n^{2\theta}}{n} \right)^p \\ & \leq (4 \cdot 3^p + 3 \cdot 2^p C_3 + 2^p + 3p) \frac{1}{n^\gamma}. \end{aligned}$$

Therefore, with $C_5 := \max(4C_3 + 1, 4 \cdot 3^p + 3 \cdot 2^p C_3 + 2^p + 3p)$ we can write

$$A(T_n, H) \leq B(T_n, H) + \frac{C_5}{n^\gamma} A(T_n, E) + \frac{C_5}{n^\gamma} B(T_n, E). \quad (41)$$

2.4 Reviewing the second case

As for the second case (see Subsection 5.4 in [NT13] and Subsection 3.4 in [TV13]), we have to track carefully the error terms.

Consider an inner extremal $\zeta_{k,j} \in E$ where $k \in \{1, 2, \dots, m\}$ and $j \in \{1, \dots, j_k\}$ are fixed.

Let H be an interval of second type containing $\zeta_{k,j}$. Then, by property (III), $|H| \leq 4n^{\gamma-\kappa}$. So if δ is fixed and n is large enough, then $|\zeta_{k,j} - \zeta_{k,j}(\delta)| \geq |H|$. Therefore H contains no inner extremals of U_δ and it behaves as an integral of first type with respect to $E(\delta) = U_\delta^{-1}[-1, 1]$.

We need the following notations and observations. Let $C_6 := \max(C_3 + 3^p, 3 + 3^p)$. Since $0 < p \leq 1$, and $D_\delta \geq 1$, we have $D_\delta^{1-p} \leq D_\delta$. We use (40) and $0 \leq \gamma < 1$, so if n is large, then

$$F_n^{p/2} \leq n^{-\gamma}.$$

The error terms in Lemma 10 have polynomial decay in the following sense. We use (9), (IV):

$$F_n^p + 3^p \left(\frac{n^{2\theta}}{n} \right)^p \leq C_3 n^{-\gamma} + 3^p n^{-\gamma} \leq \frac{C_6}{n^\gamma}, \quad (42)$$

$$3^p \left(\frac{n^{2\theta}}{n} \right)^p \leq \frac{C_6}{n^\gamma}, \quad (43)$$

and we also use (II-a), (II-b), so

$$D_\delta^{1-p} a(T_n, H_b) \leq D_\delta^{1-p} n^{-\gamma} \leq \frac{D_\delta}{n^\gamma}, \quad (44)$$

$$\begin{aligned} D_\delta^{1-p} F_n^{p/2} + D_\delta^{1-p} 3^p \left(\frac{n^{2\theta}}{n} \right)^p + \left(\frac{n^{2\theta}}{n} \right)^p b(T_n, H_b) & \leq D_\delta^{1-p} (1 + 3^p) \frac{1}{n^\gamma} + \frac{1}{n^\gamma} \\ & \leq \frac{D_\delta C_6}{n^\gamma} \end{aligned} \quad (45)$$

and with (9)

$$D_\delta (F_n^p + b(T_n, H_b)) \leq \frac{D_\delta C_6}{n^\gamma}. \quad (46)$$

Furthermore, by Lemma 4 and $0 < p < 1$, we have the following estimate for $t \in H$

$$\omega_{\Gamma_E}^{1-p}(e^{it}) \leq \omega_{\Gamma_{E(\delta)}}^{1-p}(e^{it}) \leq D_\delta \omega_{\Gamma_E}^{1-p}(e^{it}). \quad (47)$$

Now we start the estimate, as in pp. 147–149 in [NT13]. Using (13) and (47),

$$\begin{aligned} A(T_n, H) &\leq A(T_n q, H) + \frac{C_6}{n^\gamma} A(T_n, E) + \frac{C_6}{n^\gamma} B(T_n, E) \\ &\leq A_\delta(T_n q, H) + \frac{C_6}{n^\gamma} A(T_n, E) + \frac{C_6}{n^\gamma} B(T_n, E). \end{aligned}$$

We apply the first case for the polynomial $T_n q$ on the interval H with respect to $E(\delta)$ (see (41)), and use that $\deg T_n q \geq n$, so

$$A_\delta(T_n q, H) \leq B_\delta(T_n q, H) + \frac{C_5}{n^\gamma} A_\delta(T_n q, E(\delta)) + \frac{C_5}{n^\gamma} B_\delta(T_n q, E(\delta)) \quad (48)$$

where C_5 depends on p and C_3 only, C_3 depends on C_1 and C_2 (and γ, θ) only, C_1 and C_2 depend on θ and κ only. Therefore, C_5 is independent of δ .

We estimate the error term containing $A_\delta(.,.)$ using (16) (with the estimates (44), (45)), (47) and (13) (with the (42), (43)) as follows

$$\begin{aligned} A_\delta(T_n q, E(\delta)) &\leq A_\delta(T_n q, H) + \frac{D_\delta}{n^\gamma} A(T_n, E) + \frac{D_\delta C_6}{n^\gamma} B(T_n, E) \\ &\leq D_\delta A(T_n q, H) + \frac{D_\delta}{n^\gamma} A(T_n, E) + \frac{D_\delta C_6}{n^\gamma} B(T_n, E) \\ &\leq D_\delta A(T_n, H) + \left(\frac{D_\delta}{n^\gamma} + \frac{D_\delta C_6}{n^\gamma} \right) A(T_n, E) + \frac{2D_\delta C_6}{n^\gamma} B(T_n, E) \\ &\leq D_\delta \left(1 + \frac{C_6 + 1}{n^\gamma} \right) A(T_n, E) + \frac{2D_\delta C_6}{n^\gamma} B(T_n, E) \end{aligned}$$

We estimate the error term with $B_\delta(.,.)$ with (17) (and (46)) and (15) as follows

$$\begin{aligned} B_\delta(T_n q, E(\delta)) &\leq B_\delta(T_n q, H) + \frac{D_\delta C_6}{n^\gamma} B(T_n, E) \\ &\leq D_\delta B(T_n q, H) + \frac{D_\delta C_6}{n^\gamma} B(T_n, E) \leq D_\delta B(T_n, H) + \frac{D_\delta C_6}{n^\gamma} B(T_n, E) \\ &\leq D_\delta \left(1 + \frac{C_6}{n^\gamma} \right) B(T_n, E). \end{aligned}$$

We estimate the B_δ on the right hand side of (48) as follows

$$B_\delta(T_n q, H) \leq D_\delta B(T_n q, H) \leq D_\delta B(T_n, H).$$

Collecting these estimates together, we can write

$$\begin{aligned} A(T_n, H) &\leq D_\delta B(T_n, H) \\ &\quad + A(T_n, E) \left(\frac{C_5}{n^\gamma} D_\delta \left(1 + \frac{1+C_6}{n^\gamma} \right) + \frac{C_6}{n^\gamma} \right) \\ &\quad + B(T_n, E) \left(\frac{C_6}{n^\gamma} + \frac{2C_5 C_6 D_\delta}{n^{2\gamma}} + \frac{C_5}{n^\gamma} D_\delta \left(1 + \frac{1+C_6}{n^\gamma} \right) \right). \end{aligned}$$

Here, we estimate the coefficients of the error terms as follows with $C_7 := \max(C_5(2+C_6) + C_6, C_6 + 3C_5 C_6 + 2C_5, C_5)$

$$\begin{aligned} \frac{C_5}{n^\gamma} D_\delta \left(1 + \frac{1+C_6}{n^\gamma} \right) + \frac{C_6}{n^\gamma} &\leq \frac{C_5}{n^\gamma} D_\delta (1 + 1 + C_6) + \frac{C_6}{n^\gamma} \\ &\leq \frac{D_\delta}{n^\gamma} (C_5(2+C_6) + C_6) \leq \frac{D_\delta C_7}{n^\gamma} \end{aligned}$$

and

$$\begin{aligned} \frac{C_6}{n^\gamma} + \frac{2C_5 C_6 D_\delta}{n^{2\gamma}} + \frac{C_5}{n^\gamma} D_\delta \left(1 + \frac{1+C_6}{n^\gamma} \right) &\leq \frac{C_6}{n^\gamma} + \frac{2C_5 C_6 D_\delta}{n^\gamma} \\ &\quad + \frac{C_5}{n^\gamma} D_\delta (1 + 1 + C_6) \leq \frac{D_\delta}{n^\gamma} (C_6 + 2C_5 C_6 + C_5(2+C_6)) \leq \frac{D_\delta C_7}{n^\gamma}. \end{aligned}$$

Therefore, we have the following estimate

$$A(T_n, H) \leq D_\delta B(T_n, H) + \frac{D_\delta C_7}{n^\gamma} A(T_n, E) + \frac{D_\delta C_7}{n^\gamma} B(T_n, E). \quad (49)$$

Note that C_7 depends on C_5 and C_6 only, and these two constants are independent of δ .

2.5 Proving Theorem 1 for T-sets, then for union of finitely many intervals and Theorem 2

In this section we finish the proof of Theorem 1, following essentially Sections 5.5 and 6 in [NT13]. With Sections 2.3 and 2.4 at hand, we sum up the results for those intervals, copying the steps in Section 5.5 in [NT13]. This way, using $D_\delta \geq 1$ and $C_7 \geq C_5$, we can write

$$A(T_n, E) \leq D_\delta B(T_n, E) + 12N \frac{C_{29} D_\delta}{n^\gamma} A(T_n, E) + 12N \frac{C_{29} D_\delta}{n^\gamma} B(T_n, E).$$

Hence, Theorem 1 is proved for T-sets.

For sets consisting of union of finitely many intervals, Section 6 in [NT13] can be applied mutatis mutandis.

As regards Theorem 2, the Section 7 in [NT13] with the simple cosine substitution gives the proof, since the authors did not use there that the power is bigger than 1.

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